

# M4041 : Lecture 1

## 1. Differentiable Manifolds

Definition 1.1 Let  $M$  be a set. An  $n$ -dimensional coordinate system or chart on  $M$  is a pair  $(U, \varphi)$  consisting of a subset  $U \subset M$  and an injective map

$$\varphi: U \subset M \longrightarrow \mathbb{R}^n$$

where  $\varphi(U)$  is open in  $\mathbb{R}^n$ .

Definition 1.2 A set  $M$  is called a smooth manifold of dimension  $n$  if there exists a family of  $n$ -dimensional charts

$$\{(U_\alpha, \varphi_\alpha) \mid \alpha \in J\}$$

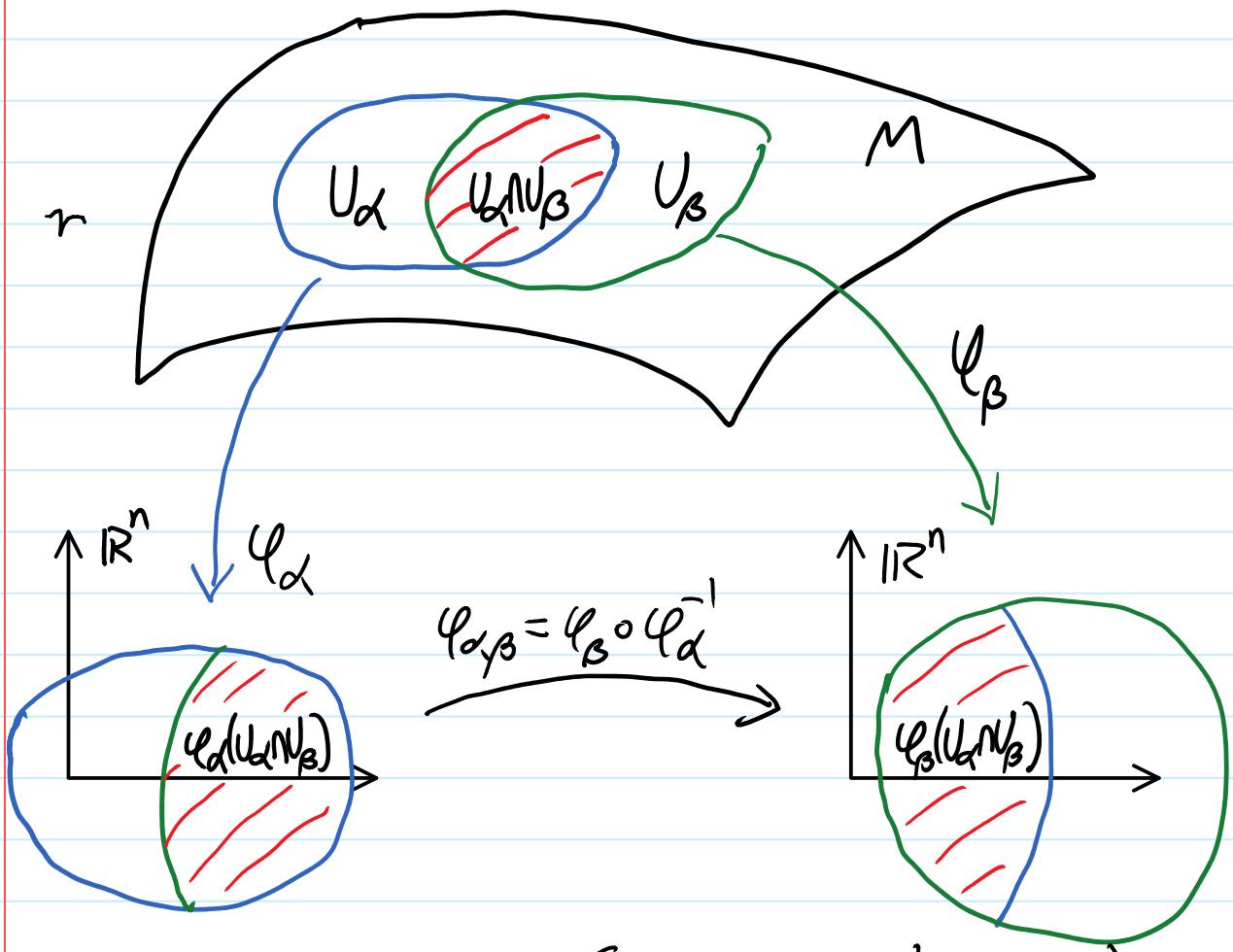
on  $M$  that satisfy :

(i)  $M = \bigcup_{\alpha \in J} U_\alpha$

(ii) if  $U_\alpha \cap U_\beta \neq \emptyset$ , then the set  $\varphi_\alpha(U_\alpha \cap U_\beta)$  is open in  $\mathbb{R}^n$  and the transition map

$$\varphi_{\alpha\beta} := \varphi_\beta \circ (\varphi_\alpha^{-1} \mid_{\varphi_\alpha(U_\alpha \cap U_\beta)}): \varphi_\alpha(U_\alpha \cap U_\beta) \rightarrow \varphi_\beta(U_\alpha \cap U_\beta)$$

is smooth. Terminology: two charts  $(U, \varphi), (V, \psi)$  satisfying  $U \cap V \neq \emptyset$  are said to overlap smoothly if  $\psi \circ \varphi^{-1} \mid_{\varphi(U \cap V)}$  is a smooth diffeomorphism.



The set of charts  $\{(U_\alpha, \varphi_\alpha) | \alpha \in J\}$  is called an atlas for  $M$ .

Remark 1.3 Atlases, in general, not unique.  
 In particular, if  $\{(U_\alpha, \varphi_\alpha) | \alpha \in J\}$  is an atlas for a smooth manifold  $M$ , and  $(V, \psi)$  is a chart on  $M$  such that

$$\psi \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap V) \rightarrow \psi(U_\alpha \cap V)$$

is a smooth diffeomorphism, then

$$\{(U_\alpha, \varphi_\alpha) | \alpha \in J\} \cup \{(V, \psi)\}$$

is also an atlas for  $M$ .

End of Remark.

Proposition 1.4 Every atlas for a smooth manifold is contained in a maximal atlas.

Remark 1.5 The charts in an atlas  $\{(U_\alpha, \varphi_\alpha) | \alpha \in J\}$  can be used to define a topology on  $M$  in the following fashion:

A subset  $V \subset M$  is called open if for each  $x \in V$  there exists a  $\alpha \in J$  such that  $x \in U_\alpha$  and  $\varphi_\alpha(U_\alpha \cap V)$  is open in  $\mathbb{R}^n$ .

### Examples 1.6

(i)  $\mathbb{R}^n$

(ii)  $S^n = \{x \in \mathbb{R}^{n+1} \mid |x|=1\}$

Let  $U = S^n \setminus \{(0, 0, \dots, 0, 1)\}$

and define  $(\tilde{x} = (x^1, x^2, \dots, x^n))$

$$\varphi: U \rightarrow \varphi(U) \subset \mathbb{R}^n: x = (\tilde{x}, x^{n+1}) \mapsto \frac{\tilde{x}}{1-x^{n+1}}$$

Then it is not difficult to show that  $\varphi$  is injective and  $\varphi(U)$  is open.

Let

$$V = S^n \setminus \{(0, \dots, 0, -1)\}$$

and define

$$\varphi: V \rightarrow \varphi(V) \subset \mathbb{R}^n: (\tilde{x}, x^{n+1}) \mapsto \frac{\tilde{x}}{1 + x^{n+1}}$$

Then it is not difficult to show that  $\varphi$  is injective,  $\varphi(V)$  is open and

$$\varphi \circ \varphi^{-1}(x) = \frac{x}{|x|^2}$$

defines a  $C^\infty$  diffeomorphism. This shows that  $S^n$  is a smooth manifold.

### Example 1.6

Suppose that  $M$  is a manifold and  $N \subset M$  is open. Let  $\mathcal{Q} = \{(U_\alpha, \varphi_\alpha) \mid \alpha \in J\}$  be an atlas for  $M$ . Then

$$\mathcal{B} = \{(U_\alpha \cap N, \varphi_\alpha|_{U_\alpha \cap N}) \mid (U_\alpha, \varphi_\alpha) \in \mathcal{J}, U_\alpha \cap N \neq \emptyset\}$$

is an atlas for  $N$  and gives  $N$  the structure of a smooth manifold. We say that  $N$  is an open submanifold of  $M$ .

### Example 1.7

Let  $M$  and  $N$  be manifolds and suppose that  $(U, \varphi)$  and  $(V, \psi)$  are charts of  $M$  and  $N$ , respectively. Then

$$\varphi \times \psi : U \times V \subset M \times N \rightarrow \varphi(U) \times \psi(V) \subset \mathbb{R}^m \times \mathbb{R}^n$$

defines a chart of  $M \times N$ . If  $\mathcal{A}$  and  $\mathcal{B}$  are atlases on  $M$  and  $N$ , respectively, it follows that

$$\mathcal{A} \times \mathcal{B} = \{(U \times V, \varphi \times \psi) \mid (U, \varphi) \in \mathcal{A}, (V, \psi) \in \mathcal{B}\}$$

defines a atlas on  $M$ , which in turn, implies that  $M \times N$  is a manifold known as the product manifold of  $M$  and  $N$ .

Definition 1.8 Let  $M$  be an  $n$ -dimensional smooth manifold. A subset  $S \subset M$  is called a submanifold of dimension  $m$ ,  $1 \leq m \leq n$ , if for each  $x \in S$  there exists chart  $(U, \varphi)$  such that

$$(i) \quad x \in U,$$

and

$$(ii) \quad \varphi(U \cap S) = \varphi(U) \cap \mathbb{R}^m \times \underbrace{\{0, \dots, 0\}}_{n-m}.$$

Remark 1.9 Let  $M$  be a smooth  $n$ -dimensional manifold with atlas  $\mathcal{A} = \{(U_\alpha, \varphi_\alpha) \mid \alpha \in J\}$ . If  $S \subset M$  is an  $m$ -dimensional submanifold, then the atlas

$$\mathcal{Q}_S = \left\{ (U_\alpha \cap S, \varphi_\alpha|_{U_\alpha \cap S}) \mid \alpha \in J_S \right\}$$

where

$$J_S = \left\{ \alpha \in J \mid (U_\alpha, \varphi_\alpha) \text{ satisfies property (ii) from Def. 1.7} \right\}$$

gives  $S$  the structure of a smooth  $m$ -dimensional manifold.

### Exercise 1.10

Let

$$C = \left\{ (x, 0) \mid x \in \mathbb{R}^2 \text{ and } \|x\| = 1 \right\}$$

and

$$S^2 = \left\{ x \in \mathbb{R}^3 \mid \|x\| = 1 \right\}.$$

Show that  $C$  is a submanifold of  $S^2$ .

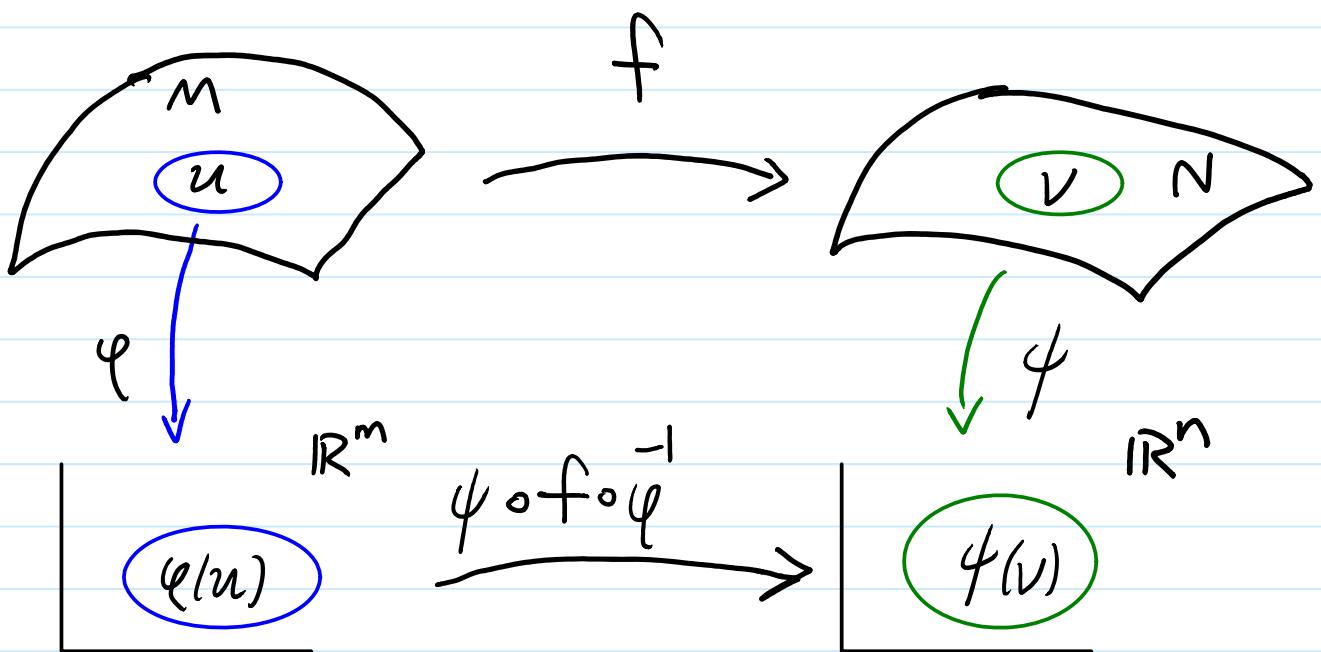
## 2. Smooth Maps

Definition 2.1 A map

$$f: M \longrightarrow N$$

between two manifolds  $M$  and  $N$  is smooth if  $\varphi \circ f \circ \varphi^{-1}$  is smooth for

all charts  $(\varphi, U)$  of  $M$  and  $(\psi, V)$  on  $N$  with  $f(U) \subset V$ .



The set of all smooth maps from  $M$  to  $N$  is denoted by  $\underline{C^\infty(M, N)}$ , or simply  $\underline{C^\infty(M)}$  if  $N = \mathbb{R}$ .

### Local expressions

The map

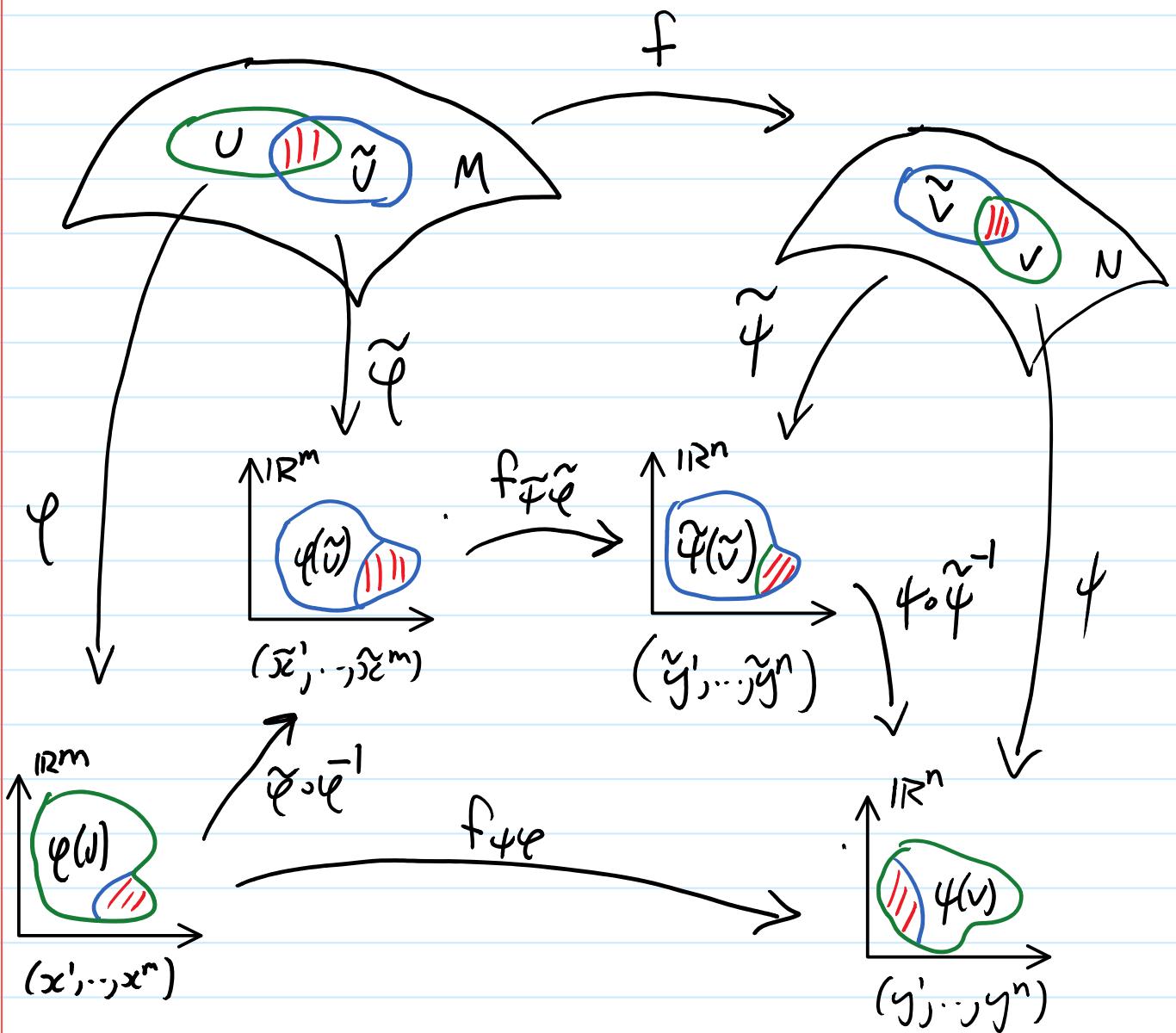
$$f_{\varphi\psi} := \psi \circ f \circ \varphi^{-1} \in C^\infty(\varphi(U), \psi(V))$$

is the local version of  $f$  in the charts  $(U, \varphi)$  and  $(V, \psi)$ .

If  $N = \mathbb{R}$ , then we can take  $\psi: \mathbb{R} \rightarrow \mathbb{R}$  to be the identity map and the local version of  $f \in C^\infty(M)$  in the chart  $(U, \varphi)$  is

$$f_\varphi := f \circ \varphi^{-1} \in C^\infty(\varphi(U))$$

## Coordinate Transformation Law



$$f_{\psi \varphi} = \psi \circ \tilde{\varphi}^{-1} \circ f \circ \tilde{\varphi} \circ \tilde{\varphi}^{-1} \circ \tilde{\varphi} \circ \tilde{\varphi}^{-1}$$

This is often informally written as

$$f(x) = y(f(\tilde{x}))$$

where

$$\tilde{x} = \tilde{\varphi}(x) := \tilde{\varphi} \circ \varphi^{-1}(x)$$

$$y = y(\tilde{y}) := \psi \circ \tilde{\varphi}^{-1}(\tilde{y})$$

## Example 2.2

- (i) IF  $f: \mathbb{R}^m \rightarrow \mathbb{R}^n$  is smooth in the usual sense , then it is also smooth in the new sense .
- (ii) The identity map  $\text{id}: M \rightarrow M : p \mapsto p$  is smooth.
- (iii) The map

$$f: S^2 \subset \mathbb{R}^3 \rightarrow \mathbb{R} : x \mapsto x \cdot \hat{k}$$

is smooth (  $\text{Hw: Show this}$  )

## Remark 2.3

Smoothness is a local concept, that is , it is not difficult to show that

$f: M \rightarrow N$  is smooth if and only if

for each  $p \in M$  there exists an open neighborhood  $U$  of  $p$  such that  $f|U$  is smooth .

Definition 2.4 A diffeomorphism is a smooth bijective map  $f: M \rightarrow N$  such that  $f^{-1}: N \rightarrow M$  is also smooth.

If there exists a diffeomorphism between  $M$  and  $N$ , then we say that they are diffeomorphic.

Example 2.5

(i)  $f: S^1 \subset \mathbb{C}^2 \rightarrow S^1 \subset \mathbb{C}^2 : z \rightarrow \bar{z}$

is a diffeomorphism.

(ii)  $f: \mathbb{R} \rightarrow \mathbb{R} : x \rightarrow x^3$

is smooth and bijective but is not a diffeomorphism.

Lemma 2.6 (Cut-off functions)

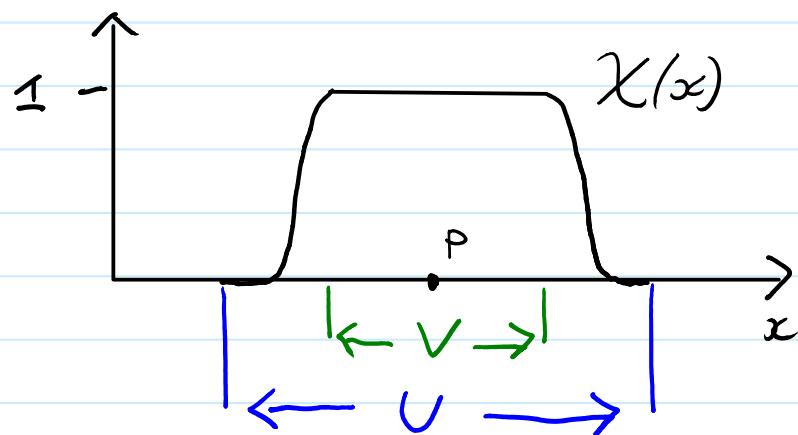
Let  $U$  be an open neighborhood of a point  $p \in M$ . Then there exist a smooth function  $\chi \in C^\infty(M)$  and a open neighborhood  $V$  of  $p$  with  $V \subset U$  such that

(i)  $0 \leq \chi \leq 1 \text{ in } M,$

(ii)  $\chi|_V = 1, \text{ and}$

(iii)  $\text{supp } \chi \subset U$

A 1-dimensional example look like



Definition 2.7 A smooth manifold  $M$ :

- (i) is Hausdorff if for any  $x, y \in M$ ,  $x \neq y$ , there exists open neighbourhoods  $U_x$  and  $U_y$  of  $x$  and  $y$  such that  $U_x \cap U_y = \emptyset$ .
- (ii) has a countable basis if  $M$  can be covered by a finite number of coordinate charts.

## Lemma 2.8 (Partitions of unity)

Let  $M$  be a smooth Hausdorff manifold with a countable basis. Then there exists a countable open cover  $\{U_\alpha\}_{\alpha \in J}$  of  $M$  by

coordinate charts  $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in J}$  and a countable collection of smooth functions  $\{\chi_\alpha\}_{\alpha \in J} \subset C^\infty(M)$

satisfying:

a)  $0 \leq \chi_\alpha \leq 1 \quad \forall \alpha \in J,$

b)  $\text{supp}(\chi_\alpha) \subset U_\alpha \quad \forall \alpha \in J,$

c)  $\sum_{\alpha \in J} \chi_\alpha = 1, \text{ and}$

d) for each  $p \in M$ , there exists an open neighbourhood  $U_p$  of  $p$  such that the set

$$\{\alpha \in J \mid \text{supp } \chi_\alpha \cap U_p \neq \emptyset\}$$

is finite.

Terminology The collection of functions

$\{\chi_\alpha\}_{\alpha \in J}$  are known as a smooth partition of unity subordinate to the open cover

$\{U_\alpha\}_{\alpha \in J}$  of  $M$ .